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Using Minimal Spanning Trees To Compare the Reliability of Network Topologies

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Introduction

The original purpose of this project was to learn some methods in graph theory and then apply them to the study of network reliability. The project proceeded far enough to produce some original material in two areas. The first area is the extensive study of braided networks, which includes a comparison of braided networks and double-ring networks. These two types of networks are described in later sections. The second area is a study of link redundancy versus path redundancy as a means of achieving reliability. These two concepts are also explained later.

In response to the increased usage of distributed computer systems, interest in network analysis techniques has greatly expanded. In a fault-tolerant network, interest is often focused on maintaining an operational path between each pair of nodes in the network, that is, ensuring that all nodes are connected, despite failure of some of the connecting links. The probability that a network is connected is known as all-terminal reliability, which is equivalent to finding the probability that the network contains at least one minimal spanning tree.

The important role that networks play in fault-tolerant systems mandates that techniques and tools be developed to aid in the analysis process. However, the development of effective and efficient techniques is hindered by the inherent computational difficulty in solving network problems. In fact, Ball has shown that network reliability problems are at least as difficult to solve as computationally hard, NP-Complete problems such as the famous traveling salesman problem (ref. 1). Consequently, most available analysis techniques primarily consist of approximation and bounding methods.

Presented in this paper are some basic combinatorial techniques from graph theory that are useful in the reliability analysis of networks. The basic assumption throughout this study is that a network is fully operational as long as it remains connected; and, the purpose of applying elementary techniques is simply to consider the concept of connectivity unencumbered by sophisticated mathematical methods and the special characteristics of individual systems. The methods presented are exact with the exception of approximations used in solving fault trees. That is, the formulas for the probability of network failure use no approximations. All the techniques used are standard procedures which can be found in textbooks (Swamy and Thulasiraman (ref. 2), Christofides (ref. 3), and Prather (ref. 4)).

Given a set of nodes and links, a variety of architectures can be configured. Although the same number of nodes and links are used, certain configurations can tolerate more link failures while preserving connectivity; hence, different architectures can have different reliabilities. The methods presented in this paper allow comparisons based on reliability to be made among different architectures. The study concentrates on the comparison of two distinct network architectures, the double-ring network and the braided network. The ring architectures are popular, and the braided network has local interest at Langley Research Center.

In the next section, the combinatorial approach to computing the reliability of networks is described. First, limitations of combinatorial methods are presented, followed by a discussion of minimal spanning trees and of the fact that, in general, reliability is not directly related to the number of minimal spanning trees in a network. The use of a fault tree program to compute the reliability of networks by means of minimal spanning trees is also discussed.

The next four sections cover the work performed. There is a comparison of double-ring and braided networks where only the links fail, followed by a comparison of the two configurations where both links and nodes fail. Also dealt with are directed links (where each link can only carry a message one direction) and links that fail in a faulty manner as opposed to being either operative or inoperative. In this case, a failed link transmits incorrect messages, and system survival depends on each node being connected by a majority of good links. The final section gives some concluding observations.

Synopsis of Combinatorial Methods

As mentioned in the introduction, developing efficient algorithms for solving network reliability problems has been proven difficult. A well-known disadvantage to using exact combinatorial methods for computing the reliability of networks is that the computational burden increases dramatically with the size of the network and quickly overwhelms even the most generous computational resources. Nevertheless, developing these exact methods has value. Many existing approximation methods must still consider the connectedness of the network, and exact methods have produced useful algorithms for determining connectedness. Other mathematical techniques, such as partitioning and conditional probabilities, originally used for exact solutions, might also be adapted for use in the more sophisticated approximation methods. Moreover, performing the simple calculations on modest networks gives the analyst a feeling for the problem along with an appreciation for the difficulty in solving it.

Limitations

The methods considered in this study assume that a network survives or remains operational as long as there are enough good components to remain connected. This definition naturally lends itself to study by combinatorial methods. This paper first examines networks that experience link failures but no node failures and then deals with models that include both node and link failures. These models allow networks to survive the failure of a large number of components. In reality, the decision algorithms of an operating system are likely to break down after only a few components have failed since there may be too much overhead used in storing and running decision algorithms that can handle all the contingencies associated with numerous failures. Within this context, the reliability results presented in this paper are likely to be optimistic; yet, valid, qualitative comparisons can still be made between real-world network architectures like the double-ring and braided networks.

The reliability aspect of the network problem is considered without regard for performance issues. Consequently, a network that appears highly reliable might have to be rejected, in practice, because its complex topology does not permit efficient routing algorithms. There are also some disadvantages inherent in using combinatorial methods (either exact or approximate) for computing system reliability. Combinatorial methods cannot capture the dynamic features of a system, so there can be no consideration of fault latency or path-regrowing time. Combinatorial methods also underestimate the probability of system failure because they do not take into account the problems created by the time lag of a system recognizing and replacing failed components. Nevertheless, combinatorial methods provide a means of making some important observations about basic network reliability.

Minimal Spanning Trees

The concept of minimal spanning trees is fundamental to the study of connectedness. For a given graph, a minimal spanning tree is a subset of the links that allows all nodes in the graph to be connected, with the property that no subset of the minimal spanning tree's links exists that forms a connected graph. That is, failure of any single link in the minimal spanning tree means that set of links no longer constitutes a minimal spanning tree, i.e., it is no longer possible to reach every node in the graph. Given a graph with n nodes, each minimal spanning tree for that graph will have $(n - 1)$ links. Figure 1 is an example of a simple network; figure 2 depicts the set of all minimal spanning trees for the network in figure 1.

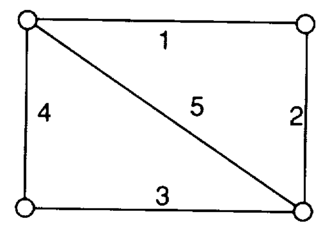


Figure 1. A simple network with bidirectional links.

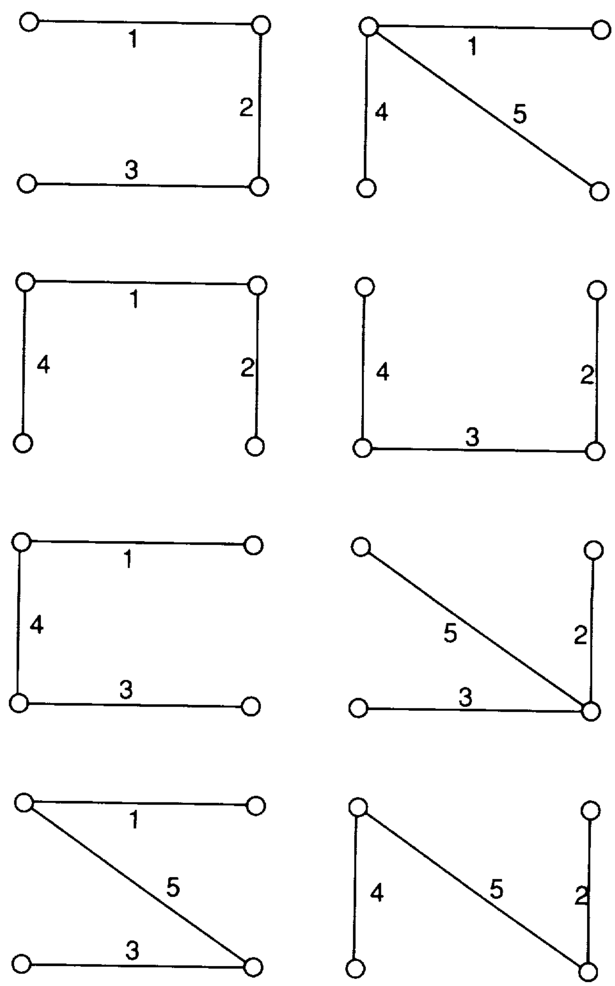
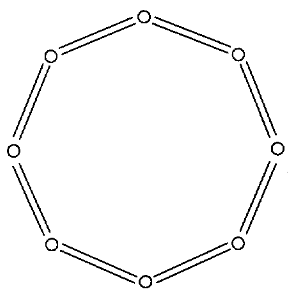


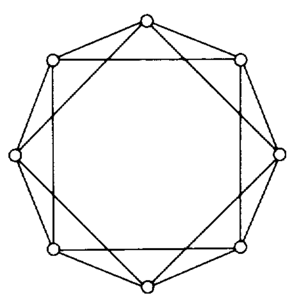
Figure 2. The set of minimal spanning trees for the network in figure 1.

The important concept to remember is that as long as all the links making up at least one of the minimal spanning trees remain operational, the network is connected and, hence, assumed operational. Thus, the probability that the network fails is equivalent to the probability that no operational minimal spanning tree exists. Theoretically, then, computing the reliability of a network is quite simple. However, even for networks with very few nodes (<10), the number of spanning trees can be very large, which makes storing the trees and finding the probability that no operational spanning trees exists a computationally demanding task.

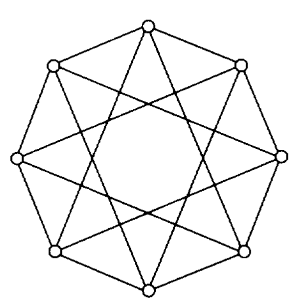
The number of minimal spanning trees for a network can be computed without actually generating the list of trees (ref. 3). Figure 3 shows a double-ring, a two-braid, and a three-braid network. In the two-braid network, the inner links connect every second node instead of adjacent nodes. In the three-braid network the inner links connect every third node.



(a) Double ring.



(b) Two braid.



(c) Three braid.

Figure 3. Three network configurations considered in present paper.

Table I contains the number of minimal spanning trees for each of the configurations above as the number of nodes varies from 5 to 11.

Table I. Number of Minimal Spanning Trees for Double-Ring, Two-Braid, and Three-Braid Network Configurations

Number of nodes	Number of minimal spanning trees		
	Double ring	Two braid	Three braid
5	80	125	
6	192	384	
7	448	1 183	1 183
8	1 024	3 528	4 096
9	2 304	10 404	12 321
10	5 120	30 250	40 500
11	11 264	87 131	130 691

Although two networks have equal numbers of nodes and links, the links can be arranged in a variety of architectures such that the two networks have different numbers of minimal spanning trees, as in figures 4 and 5. Since only one minimal spanning tree must remain intact in order for a network to be connected, it seems logical to assume that the network with the larger number of minimal spanning trees will be the more reliable of the two. Network reliability would appear to be proportional to the number of spanning trees.

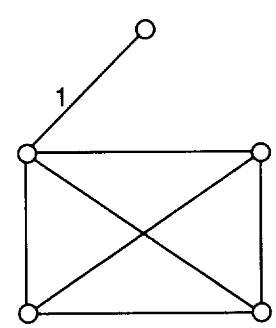


Figure 4. A network with 16 minimal spanning trees.

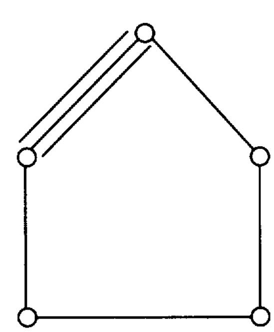


Figure 5. A network with 13 minimal spanning trees.

The network in figure 4 has 16 minimal spanning trees and the network in figure 5 has 13 minimal spanning trees. Based on the number of spanning trees, it would appear that the network in figure 4 is more reliable than the network in figure 5. However, a single link failure, failure of link 1, will isolate a node in figure 4 whereas it takes failure of at least two links to isolate a node in figure 5. Thus, the network in figure 5 is more reliable than the network in figure 4 since its cutset (the number of links it takes to isolate a node) is larger than the cutset of the network in figure 4. Computations reveal that when the probability of link failure is 10^{-2} the probability of failure for the network in figure 4 is 1.0004×10^{-2} and the probability of failure for the network in figure 5 is 5.9206×10^{-4}.

Therefore, network reliability comparisons cannot be made based on the number of minimal spanning trees. A monotonic relationship between reliability and the number of minimal spanning trees may hold for symmetric networks, examples of which will be discussed later, but there is no proven result about a general monotonic relationship. This counter-example is unfortunate since the method of computing the number of minimal spanning trees is easy, while obtaining all the minimal spanning trees is difficult.

Fault Trees

As discussed earlier, computing the probability that no minimal spanning tree will survive is equivalent to computing the probability that the network will fail. That is, given the complete list of minimal spanning trees for a network and the probability of failure for each link in the network, combinatorial methods can be applied to compute the probability of network failure. Fault tree analysis is a convenient method for computing combinatorial probabilities. To demonstrate the role of fault trees in computing network reliability, consider the example of figure 6.

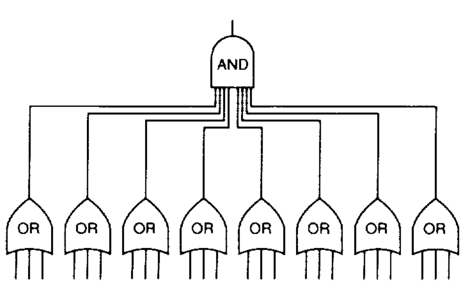


Figure 6. Fault tree to compute the probability of network failure for the network depicted in figure 1.

Figure 6 shows the fault tree for computing the probability of failure of the network shown in figure 1. The basic events, 1, 2, 3, 4, and 5, represent failures of the individual links. Each OR gate corresponds to a minimal spanning tree. Failure of one of the inputs to an OR gate eliminates the associated spanning tree as a possibility for preserving network connectivity. The top gate, representing network failure, is an AND gate since the network is not connected if all the minimal spanning trees are eliminated.

Even for this small example, manually calculating the probability of the top event is tedious. And, as seen in table I, the number of spanning trees in a modest size network can be extremely large, which would make manual calculation of reliability infeasible. Fortunately, computer programs have been created to effectively perform fault tree analysis. The Fault Tree Compiler program, developed at Langley Research Center by Butler and Martensen (ref. 5), employs an efficient algorithm for solving fault trees and is used throughout this study to provide numerical results.

Reliability When Bidirectional Links Fail

A common assumption in network analysis is that the links are subject to random failure and that the nodes do not fail. From a practical viewpoint, considering only link failure implies that all processing elements (nodes) in a network are functional but the communication channels (links) between those processing elements are subject to failure. A bidirectional link allows communication to flow between two nodes in either direction. Failure of the links was studied by comparing the reliability of networks with double-ring and braid configurations.

For this section and the next, the first step in computing reliability was to generate the list of minimal spanning trees for the networks. Because of the combinatorial complexity of generating these lists, the accuracy of the results was checked by implementing two methods of tree generation (refs. 2 and 3). An additional check was made by using the previously mentioned formula (ref. 3) for computing the number of minimal spanning trees without generating them. In all the cases, the two lists of minimal spanning trees were identical, and the number of trees generated was correct. After generating the minimal spanning trees, a fault tree based on the spanning trees was created in an acceptable input format and submitted to the Fault Tree Compiler program.

Bidirectional networks configured as braids and double rings containing between five and nine nodes inclusive were studied. Limited computing capacity restricted network size to a maximum of nine nodes.

Probability of link failure was varied from 10^{-1} to 10^{-5}. Table II contains the reliability results from the study of double-ring networks and table III contains the reliability results for the two-braid networks.

In each case, the two-braid configuration is more reliable than the corresponding double ring. As the probability of failure of the links decreases by one order of magnitude, the probability of system failure decreases by four orders of magnitude regardless of the configuration or number of nodes in the network. Increasing the number of nodes in the network slightly decreased the reliability for both architectures. This effect seems reasonable since it is more difficult to keep a system fully connected as the number of nodes increases.

Reliability When Nodes and Bidirectional Links Fail

When both the links and the nodes are allowed to fail, two assumptions are made for a network to remain operational: (1) at least a strict majority of the nodes must be working and (2) all working nodes must be connected. These assumptions are made because they are simple and reasonable. A strict majority of good nodes can outvote the corrupt data from the failed nodes. The good nodes must be connected in order to work together. To study the effects of node and link failures, consider the braided six-plex pictured in figure 7.

Success for this six-plex network requires that no more than two nodes fail, since there must be at least four working nodes to maintain a majority. The key to computing reliability for networks with both node and link failures is to consider a partition of the sample space based on the number of node failures. Figure 8 shows the five possible successful architectures for the network in figure 7.

Table II. Reliability Results for Double-Ring Networks

[L = Probability of link failure]

Number of nodes	Probability of system failure				
	$L = 10^{-1}$	$L = 10^{-2}$	$L = 10^{-3}$	$L = 10^{-4}$	$L = 10^{-5}$
5	0.980150×10^{-3}	0.999800×10^{-7}	0.999998×10^{-11}	0.999410×10^{-15}	0.999941×10^{-19}
6	1.46045	1.49960	1.49996	1.50000	1.50000
7	2.03104	2.09912	2.09999	2.10000	2.10000
8	2.69008	2.79887	2.79999	2.80000	2.80000
9	3.43573	3.59832	3.59998	3.60000	3.60000

Table III. Reliability Results for Two-Braid Networks

[L = Probability of link failure]

Number of nodes	Probability of system failure				
	$L = 10^{-1}$	$L = 10^{-2}$	$L = 10^{-3}$	$L = 10^{-4}$	$L = 10^{-5}$
5	5.07758×10^{-4}	5.00098×10^{-8}	5.00001×10^{-12}	5.00000×10^{-16}	5.00000×10^{-20}
6	6.13153	6.00157	6.00002	6.00000	6.00000
7	7.17702	7.00205	7.00002	7.00000	7.00000
8	8.13132	8.00272	7.99948	7.99999	8.00000
9	9.29784	9.00018	8.99919	8.99999	9.00000

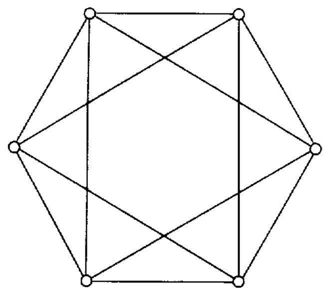


Figure 7. Two-braided six-plex.

Figure 8 depicts the possible architectures for a braided six-plex when a strict minority of nodes fail and the links attached to a failed node are no longer operational (and are removed from the diagram). There is only one architecture for zero failed nodes and one architecture for one failed node. The network is complex enough that there are three architectures possible when two nodes fail. The probability of each of these three architectures, given two failed nodes, is simply a ratio

$$P\{\text{architecture I, given two failed nodes}\} = 6/15 = 2/5$$

$$P\{\text{architecture II, given two failed nodes}\} = 6/15 = 2/5$$

$$P\{\text{architecture III, given two failed nodes}\} = 3/15 = 1/5$$

If the probability of a node failure is defined to be v, the probability that n out of six nodes will fail is given by the combinatorial formula

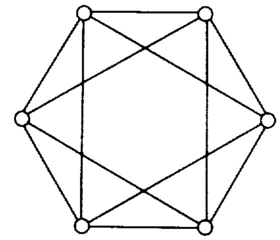
$$\binom{6}{n} v^n (1-v)^{6-n}$$

where

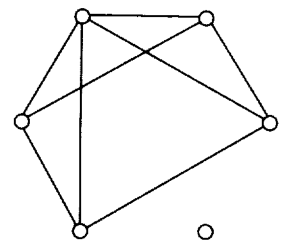
$$\binom{6}{n} = \frac{6!}{n!(6-n)!}$$

The formula for computing network failure F using the partition on the number of node failures is

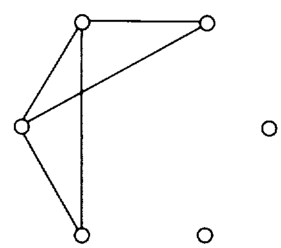
$$\begin{aligned} P\{F\} = {} & P\{F, \text{ given zero fail}\}\, P\{\text{zero fail}\} \\ & + P\{F, \text{ given one fails}\}\, P\{\text{one fails}\} \\ & + P\{\text{arch. I fails}\}\, P\{\text{arch. I, given two fail}\}\, P\{\text{two fail}\} \\ & + P\{\text{arch. II fails}\}\, P\{\text{arch. II, given two fail}\}\, P\{\text{two fail}\} \\ & + P\{\text{arch. III fails}\}\, P\{\text{arch. III, given two fail}\}\, P\{\text{two fail}\} \\ & + P\{\text{three or more fail}\} \end{aligned}$$



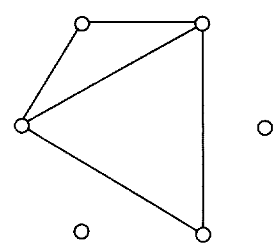
(a) Zero failed nodes.



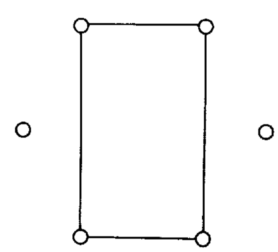
(b) One failed node.



(c) Two failed nodes: architecture I. Adjacent nodes fail (six cases).



(d) Two failed nodes: architecture II. Failed nodes separated by one node (six cases).



(e) Two failed nodes: architecture III. Failed nodes separated by two nodes (three cases).

Figure 8. The five possible architectures for a two-braided six-plex when links are removed because of node failures.

Figure 9 shows the probability of network failure for braids containing five to nine nodes, and figure 10 shows corresponding double-ring computations. In this study, probability of link failure was held constant at 10^{-2}, and probability of node failure ranged from 10^{-1} to 10^{-4}. The data for figures 9 and 10 are presented in the appendix.

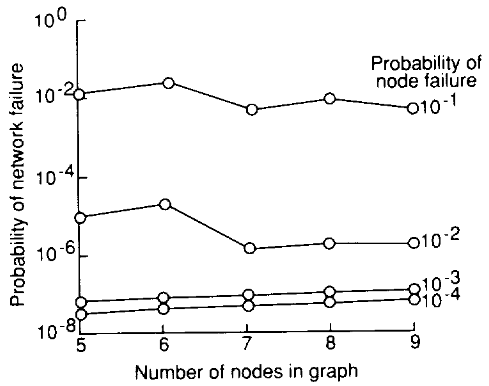


Figure 9. Probability of network failure for two-braid networks.

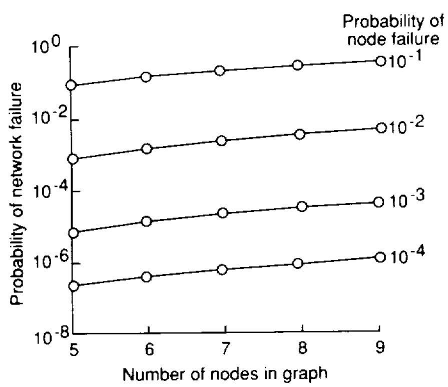


Figure 10. Probability of network failure for double-ring networks.

The double-ring networks behave as expected; as the number of nodes increases, the probability of system failure slightly increases. The behavior of the two-braid networks, especially when the probability of node failure is large, is not as easily explained. Part of the variation can be attributed to requiring a strict majority of nodes to be operational for system survival. For instance, in the six-node networks, four nodes must be working for the system to survive. Four nodes are also needed for survival of seven-node networks. Thus, it is easier to have the strict majority of nodes working in networks with an odd number of nodes; hence, the probability of system failure decreases at these points. This decrease is more profound when the probability of node failure is relatively high, since node failures would play a more prominent role in causing system failure. This strict majority phenomenon does not have much influence on the double-ring networks because a double-ring network will almost always fail from node failures long before a majority of nodes fail. For example, two node failures will cause the failure of a double-ring network unless the two failed nodes are adjacent.

Reliability When Unidirectional Links Fail

For the systems discussed in this section, a link consists of a transmitter, a line, and a receiver. Messages are sent in only one direction on each link. The criterion for network survival is that all the nodes be able to send and receive messages to each other. Unlike the minimal spanning sets for bidirectional networks, minimal spanning sets for unidirectional networks may contain minimal spanning trees with different numbers of elements. Consider the network in figure 11 and two of its minimal spanning trees pictured in figure 12.

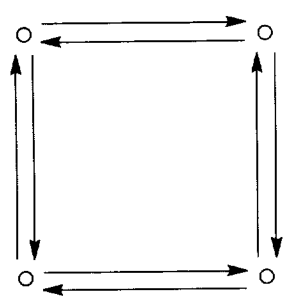


Figure 11. A network with unidirectional links.

A method for generating the list of spanning trees for a unidirectional network is to create a relational matrix from the description of the network and then compute reachability for each node in the network (ref. 4). Evaluating a fault tree patterned after these spanning sets, just as with the minimal spanning trees of previous sections, will result in network reliability figures.

This study of unidirectional networks considered two-braid networks and double-ring networks containing five to seven nodes inclusive. Figure 13 depicts four link arrangements in order of decreasing reliability. The most reliable network is the corotational double ring. The next most reliable is the corotational two braid, followed by the counterrotational two braid. The least reliable is the counterrotational double ring. Table IV gives the numerical results for the reliability of each of these architectures.

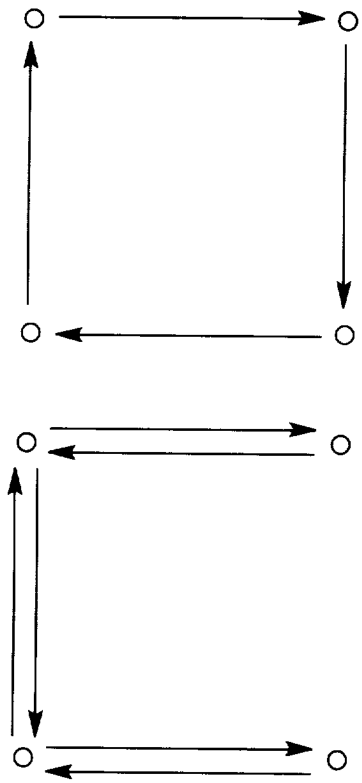
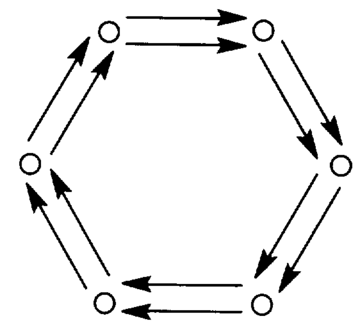


Figure 12. Two minimal spanning trees for the network in figure 11.

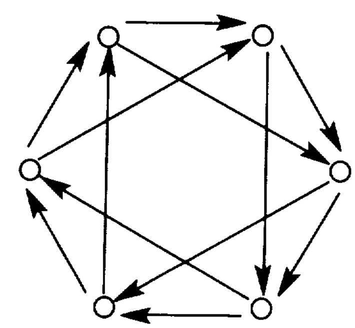
As seen in the study of bidirectional links, increasing the number of nodes slightly increased the probability of failure. Among the configurations with the same number of nodes, the corotational ring configuration is the most reliable. Differences between the corotational and counterrotational braids were minute. And, in general, there was less than one order of magnitude difference in reliability among any of the networks.

Table IV. Reliability Results for Networks With Unidirectional Links

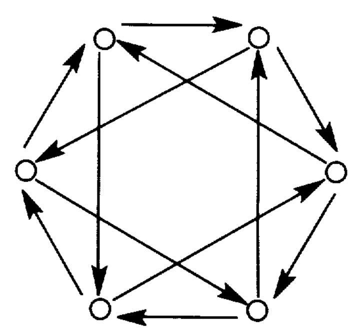
Number of nodes	Network configuration	Probability of system failure
5	Corotational ring	0.499900×10^{-3}
	Counterrotational ring	1.94060
	Corotational braid	0.989755
	Counterrotational braid	0.989755
6	Corotational ring	0.599850×10^{-3}
	Counterrotational ring	2.88194
	Corotational braid	1.18947
	Counterrotational braid	1.19509
7	Corotational ring	0.699790×10^{-3}
	Counterrotational ring	3.99465
	Corotational braid	1.38538
	Counterrotational braid	1.39843



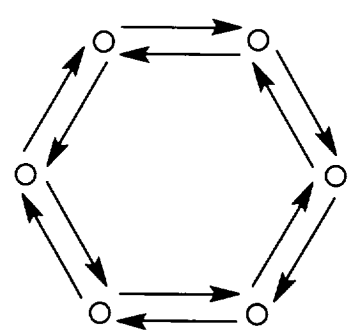
Corotational double-ring



Corotational two-braid



Counterrotational two-braid



Counterrotational double-ring

Figure 13. Four unidirectional link configurations.

Reliability With Faulty Links in Bidirectional Networks

In the previous sections, links were viewed as either operative (sending correct messages) or, through failing, inoperative (failing to send any message). This section considers faulty links, which are links that fail by transmitting incorrect messages. In this case, the assumption is that system survival depends on the connectivity of the nodes and, additionally, on each node having a majority of good links. The idea of a faulty link, rather than absolute failure, may more realistically model the behavior of communication links.

In fault-tolerant computer systems, redundancy is a common technique used to achieve higher reliability. In networks, redundancy can be implemented on different levels. This section compares two methods of providing redundancy: link redundancy and path redundancy.

In link redundancy the network is connected by a single path, where each segment of the path consists of redundant links. For example, if each segment consists of three links, then the segment fails if two or three of the links fail. If the probability of link failure is p, then the probability of segment failure s is

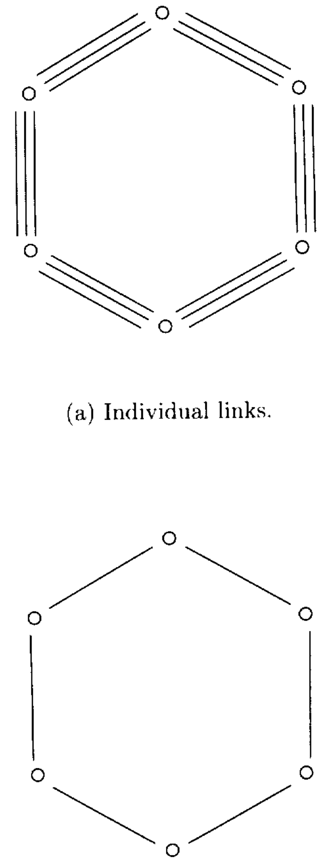
$$s = 3p^2(1-p) + p^3$$

Figure 14(a) displays a ring where each segment between nodes consists of three links and each link has probability of failure p. Figure 14(b) displays the same ring, indicating just the segments where each segment has probability of failure $s = 3p^2(1-p)+p^3$. Since this network fails if two or more segments fail, the probability of network failure is

$$\begin{aligned} P\{\text{network failure}\} &= 1 - P\{\text{zero failed segments}\} \\ &\quad - P\{\text{one failed segment}\} \\ &= 1 - (1-s)^6 - 6s(1-s)^5 \end{aligned}$$

where s is the probability of segment failure given above.

When using path redundancy, the links connecting two nodes can be considered as belonging to different levels. Figure 15(a) displays three levels. The links in the first level are represented by solid lines, in the second level by dotted lines, and in the third level by broken lines. The links in each level attempt to form a minimal spanning tree. Different levels can have different spanning trees. Figure 15(b) displays three different spanning trees for the links in figure 15(a).



(a) Individual links.

(b) Links combined into segments.

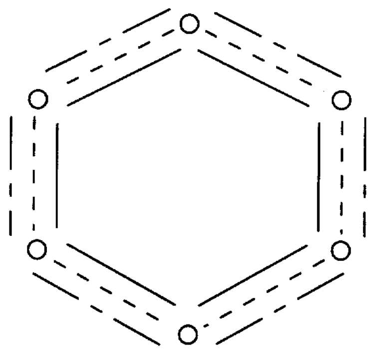
Figure 14. Link redundancy for a ring network with six nodes.

When computing the reliability using path redundancy, the first step is to calculate the probability of having a spanning tree for each level. The second step is to calculate the probability that a majority of levels have a spanning tree. For the network in figure 15(a), a ring network with three levels, a level will not have a spanning tree if two or more links fail. If the probability of a link failure is p, then the probability of a "level" failure Q is

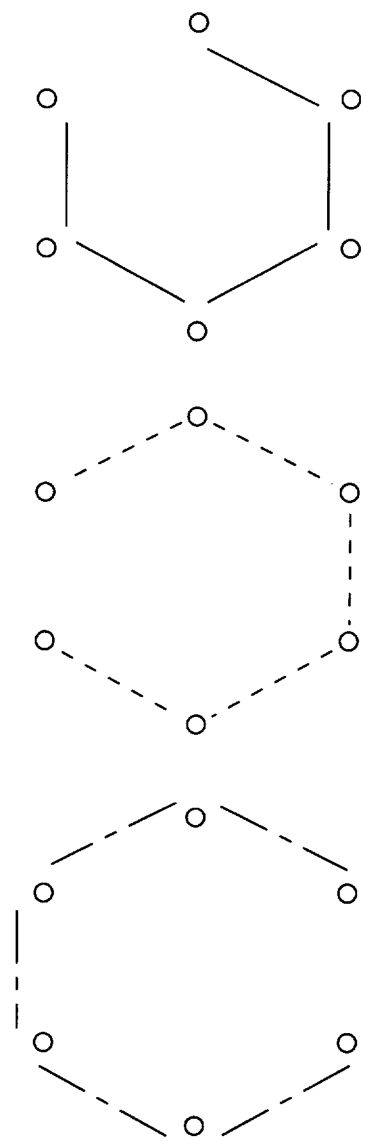
$$\begin{aligned} Q &= 1 - P\{\text{zero link failures}\} - \mathrm{P}\{\text{one link failure}\} \\ &= 1 - (1-p)^6 - 6p(1-p)^5 \end{aligned}$$

The probability of network failure is the probability that two or three levels fail, which is

$$P\{F\} = 3Q^2(1-Q) + Q^3$$



(a) Three-level six-plex.



(b) A spanning tree for each level of the three-level six-plex.

Figure 15. Illustration of path redundancy.

This issue of link redundancy versus path redundancy appears in the design of the Advanced Information Processing System (AIPS) (ref. 6). The baseline architecture of the AIPS network consists of three rings. Each ring has five nodes and two extra links connecting that ring to a computer. For the AIPS network to remain functional, at least two of the three rings must be connected. No analysis of path versus link redundancy is given, though, to justify the choice of path redundancy. In fact, the study presented in this paper is the only known comparison of these two redundancy techniques.

A comparison of link redundancy versus path redundancy was completed for two-braid networks and double-ring networks having six and seven nodes. Table V shows the probability of failure for these networks.

Table V. Probability of Network Failure Using Link and Path Redundancy

Network configuration	Probability of network failure	
	Path redundancy	Link redundancy
Triple redundancy		
Two braid, 6 nodes	1.081×10^{-14}	4.732×10^{-14}
Two braid, 7 nodes	1.471	5.520
Double ring, 6 nodes	6.746	11.83
Double ring, 7 nodes	13.22	16.56
Fivefold redundancy		
Two braid, 6 nodes	0.2162×10^{-20}	5.649×10^{-20}
Two braid, 7 nodes	0.3433	6.591
Double ring, 6 nodes	3.372	14.12
Double ring, 7 nodes	9.252	19.77

The table shows little difference in reliability between link and path redundancy. Path redundancy gives slightly higher, but not significantly higher, reliability results for each network. Therefore, the choice of redundancy method would be governed by design convenience or performance considerations.

Conclusions

Despite the disadvantages of combinatorial methods and the demanding computational requirements of network problems, using minimal spanning trees is an effective means of performing basic reliability analysis for small networks. Two network topics were examined using the minimal spanning tree approach and its generalization for directed networks. The first topic was an examination of network reliability when a link no longer delivered messages upon failure. For this case, the reliability of the popular double-ring and braided networks was studied and comparisons were made for both bidirectional and unidirectional links. The braided networks exhibited higher reliability with bidirectional links, while some of the double-ring configurations, especially the co-rotational ring, showed better reliability with unidirectional links. With both types of network configuration, the reliability of the network slightly decreased as more nodes were added.

The second topic was an examination of network reliability when a link delivered incorrect messages upon failure. For this case, the study compared the effects of link redundancy versus path redundancy on network reliability. Although path redundancy produced higher reliability in each case, the reliability was only a slight improvement over link redundancy. Thus, other considerations such as performance or engineering restrictions should impact the choice of redundancy technique. This analysis has direct application to present fault-tolerant computer systems such as the Advanced Information Processing System (AIPS) that currently implements path redundancy to achieve network reliability.

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Appendix

Reliability Results for Node and Bidirectional Link Failure

The following tables contain reliability figures for the various bidirectional network configurations which result when both links and nodes fail. The tables cover two braids and double rings containing between five and nine nodes. The probability of link failure is 10^{-2} in all the tables.

Five Nodes, Two Braid

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		10	125	5.00098×10^{-8}	
1		6	16	4.02881×10^{-6}	
2	1	3	3	2.98000×10^{-4}	1/2
2	2	3	3	2.98000×10^{-4}	1/2

Probability of node failure	Probability of system failure
10^{-1}	8.583075×10^{-3}
10^{-2}	1.038081×10^{-5}
10^{-3}	8.277992×10^{-8}
10^{-4}	5.203818×10^{-8}

Six Nodes, Two Braid

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		12	384	6.00157×10^{-8}	
1		8	45	4.04957×10^{-6}	
2	1	5	8	2.03910×10^{-4}	2/5
2	2	5	8	2.03910×10^{-4}	2/5
2	3	4	4	5.92030×10^{-4}	1/5

Probability of node failure	Probability of system failure
10^{-1}	1.587917×10^{-2}
10^{-2}	2.024682×10^{-5}
10^{-3}	1.079938×10^{-7}
10^{-4}	6.247043×10^{-8}

Seven Nodes, Two Braid

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		14	1183	7.00207×10^{-8}	
1		10	130	4.06017×10^{-6}	
2	1	7	21	2.03988×10^{-4}	1/3
2	2	7	24	1.06007×10^{-4}	1/3
2	3	6	11	4.03792×10^{-4}	1/3
3	1	5	8	2.03910×10^{-4}	1/5
3	2	4	3	1.02950×10^{-2}	2/5
3	3	3	1	2.97010×10^{-2}	1/5
3	4	4	4	5.92030×10^{-4}	1/5

Probability of node failure	Probability of system failure
10^{-1}	2.993672×10^{-3}
10^{-2}	1.493146×10^{-6}
10^{-3}	1.031453×10^{-7}
10^{-4}	7.286241×10^{-8}

Eight Nodes, Two Braid

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		16	3528	8.00272×10^{-8}	
1		12	368	4.08037×10^{-6}	
2	1	9	55	2.04988×10^{-4}	2/7
2	2	9	66	1.06029×10^{-4}	2/7
2	3	8	32	3.05946×10^{-4}	2/7
2	4	8	30	3.07848×10^{-4}	1/7
3	1	7	21	2.03988×10^{-4}	1/7
3	2	6	8	1.02019×10^{-2}	2/7
3	3	5	3	2.01921×10^{-2}	2/7
3	4	6	11	4.03792×10^{-4}	1/7
3	5	5	5	9.80150×10^{-4}	1/7

Probability of node failure	Probability of system failure
10^{-1}	5.353379×10^{-3}
10^{-2}	2.111147×10^{-6}
10^{-3}	1.185008×10^{-7}
10^{-4}	8.328735×10^{-8}

Nine Nodes, Two Braid

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		18	10 404	9.00018×10^{-8}	
1		14	1 040	4.10057×10^{-6}	
2	1	11	144	2.05998×10^{-4}	1/4
2	2	11	185	1.06049×10^{-4}	1/4
2	3	10	87	3.05968×10^{-4}	1/4
2	4	10	8	2.09955×10^{-4}	1/4
3	1	9	55	2.04988×10^{-4}	3/28
3	2	8	21	1.02019×10^{-2}	3/14
3	3	7	8	2.00999×10^{-2}	3/14
3	4	7	9	1.05899×10^{-2}	3/28
3	5	8	32	3.05946×10^{-4}	3/28
3	6	7	14	6.99656×10^{-4}	3/14
3	7	6	6	1.46045×10^{-3}	1/28
4	1	7	21	2.03988×10^{-4}	1/14
4	2	6	8	1.02019×10^{-2}	1/7
4	3	5	0	1.00000×10^{0}	1/14
4	4	5	3	2.01921×10^{-2}	2/7
4	5	4	0	1.00000×10^{0}	1/14
4	6	5	3	2.01921×10^{-2}	1/14
4	7	4	1	3.94040×10^{-2}	3/14
4	8	5	5	9.80150×10^{-4}	1/14

Probability of node failure	Probability of system failure
10^{-1}	2.47101×10^{-3}
10^{-2}	1.94486×10^{-6}
10^{-3}	1.33884×10^{-7}
10^{-4}	9.36835×10^{-8}

Five Nodes, Double Ring

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		10	80	9.99800×10^{-8}	
1		6	8	2.99970×10^{-4}	
2	1	4	4	1.99990×10^{-4}	1/2
2	2	2	0	1.00000×10^{0}	1/2

Probability of node failure	Probability of system failure
10^{-1}	4.511575×10^{-2}
10^{-2}	5.095997×10^{-4}
10^{-3}	6.589337×10^{-6}
10^{-4}	2.998600×10^{-7}

Six Nodes, Double Ring

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		12	192	1.49960×10^{-7}	
1		8	16	3.99940×10^{-4}	
2	1	6	8	2.99970×10^{-4}	2/5
2	2	4	0	1.00000×10^{0}	2/5
2	3	4	0	1.00000×10^{0}	1/5

Probability of node failure	Probability of system failure
10^{-1}	7.505259×10^{-2}
10^{-2}	9.072244×10^{-4}
10^{-3}	1.152253×10^{-5}
10^{-4}	4.797161×10^{-7}

Seven Nodes, Double Ring

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		14	448	2.09912×10^{-7}	
1		10	32	4.99900×10^{-4}	
2	1	8	16	3.99940×10^{-4}	1/3
2	2	6	0	1.00000×10^{0}	1/3
2	3	6	0	1.00000×10^{0}	1/3
3	1	6	8	2.99970×10^{-4}	1/5
3	2	4	0	1.00000×10^{0}	2/5
3	3	4	0	1.00000×10^{0}	1/5
3	4	2	0	1.00000×10^{0}	1/5

Probability of node failure	Probability of system failure
10^{-1}	1.03963×10^{-1}
10^{-2}	1.39190×10^{-3}
10^{-3}	1.76462×10^{-5}
10^{-4}	6.99457×10^{-7}

Eight Nodes, Double Ring

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		16	1024	2.79887×10^{-7}	
1		12	64	5.99850×10^{-4}	
2	1	10	32	4.99900×10^{-4}	2/7
2	2	8	0	1.00000×10^{0}	2/7
2	3	8	0	1.00000×10^{0}	2/7
2	4	8	0	1.00000×10^{0}	1/7
3	1	8	16	3.99940×10^{-4}	1/7
3	2	6	0	1.00000×10^{0}	2/7
3	3	6	0	1.00000×10^{0}	2/7
3	4	4	0	1.00000×10^{0}	1/7
3	5	4	0	1.00000×10^{0}	1/7

Probability of node failure	Probability of system failure
10^{-1}	1.39908×10^{-1}
10^{-2}	1.97465×10^{-3}
10^{-3}	2.49751×10^{-5}
10^{-4}	9.59175×10^{-7}

Nine Nodes, Double Ring

Number of failed nodes	Architecture	Number of links	Number of minimal spanning trees	Probability of system failure	Probability of architecture
0		18	2304	3.59832×10^{-7}	
1		14	128	6.99790×10^{-4}	
2	1	12	64	5.99850×10^{-4}	1/4
2	2	10	0	1.00000×10^{0}	1/4
2	3	10	0	1.00000×10^{0}	1/4
2	4	10	0	1.00000×10^{0}	1/4
3	1	10	32	4.99900×10^{-4}	3/28
3	2	8	0	1.00000×10^{0}	3/14
3	3	8	0	1.00000×10^{0}	3/14
3	4	8	0	1.00000×10^{0}	3/28
3	5	6	0	1.00000×10^{0}	3/28
3	6	6	0	1.00000×10^{0}	3/14
3	7	6	0	1.00000×10^{0}	1/28
4	1	8	16	3.99940×10^{-4}	1/14
4	2	6	0	1.00000×10^{0}	1/7
4	3	6	0	1.00000×10^{0}	1/14
4	4	6	0	1.00000×10^{0}	2/7
4	5	6	0	1.00000×10^{0}	1/14
4	6	4	0	1.00000×10^{0}	1/14
4	7	4	0	1.00000×10^{0}	3/14
4	8	2	0	1.00000×10^{0}	1/14

Probability of node failure	Probability of system failure
10^{-1}	1.77097×10^{-1}
10^{-2}	2.64726×10^{-3}
10^{-3}	3.49611×10^{-5}
10^{-4}	1.25876×10^{-6}

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16. Abstract

Graph theory methods are applied to compute the reliability of several types of networks of moderate size. The graph theory methods used are minimal spanning trees for networks with bidirectional links and the related concept of strongly connected directed graphs for networks with unidirectional links. The study compares ring networks and braided networks. It covers the case where just the links fail and the case where both links and nodes fail. It considers two failure modes for the links. For one failure mode, the link no longer carries messages. For the other failure mode, the link delivers incorrect messages. There is a description and comparison of link redundancy versus path redundancy as methods to achieve reliability. All the computations are carried out by means of a fault tree program.

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